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Primitively divergent diagrams in a scalar field with quartic self-interaction and a kappa-deformed dispersion relation

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Abstract

We obtain the dimensionally regularized primitively divergent diagrams for a scalar field model with quartic self-interaction and a kappa-deformed dispersion relation as a function of the complex dimension analytically continued to the neighborhood of all real dimensions. The result shows that the poles of those diagrams occur for odd dimensions in distinction to the poles at even dimensions of the non-deformed diagrams. Actually, the singular dimensions in the deformed case are shifted by one to the right in relation to the singular dimensions of the non-deformed case. This shifting of the poles appears as an effect of the deformation on the complex dimension plane of the dimensional regularization procedure.

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1. Introduction

The need for a fundamental length parameter has been advocated since the early days of quantum field theory, most notably by Heisenberg, who presented several arguments to add this fundamental length, say q, to the set composed of the fundamental speed c and the fundamental action $\hbar$ [1]. One of the arguments presents $1/q$ as a natural momentum cutoff for the divergences of the theory, and derives this cutoff from the hypothesis that, at lengths comparable to q, spacetime exhibits a new kind of geometry. As pointed out by Amelino-Camelia [2], the relativity principle allows the introduction of such fundamental length only if all inertial observers can agree on its value and physical interpretation, i.e. if it has an invariant character. It is well known that the combination of gravitational and quantum effects provides a fundamental length value, specifically of order 10^{-33} cm, at the Planck scale. In this context, several recent proposals have been advanced for the introduction of a characteristic scale (or the corresponding fundamental energy) (see, e.g. [2, 3] and the reviews [4]), which has also been advocated as a consequence of string theory (for a review, see [5]). Whatever the motivations to have such a fundamental length scale in a quantum field theory, we are led to investigate the new features of such theory to determine whether they open the possibility of unusual and interesting physics, or give rise to unavoidable inconsistencies. In this paper, the feature we are interested in is the role played by a fundamental length in the regularization of a quantum field theory.

The fundamental length q can be introduced in the quantum field theory in several theoretical settings. It can be taken as the deformation parameter of a deformation of the usual commutative algebra of spacetime coordinates, as is done in non-commutative spacetime field theory (see, e.g., the reviews [6]). The spacetime non-commutativity is implemented by promoting spacetime coordinates to Hermitian operators obeying relations of the form $[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}$, where the parameter $\theta^{\mu\nu}$ is antisymmetric, has a dimension of squared length and is proportional to q^2. It is found that this non-commutativity of spacetime does not necessarily eliminate all the ultraviolet divergences, and that it also gives rise to the mixing of ultraviolet and infrared divergences. Let us note that violations of Lorentz symmetry are manifest in those noncommutative theories [7]. In the original formalism of noncommutative spacetime presented by Snyder [8], the components of $\theta^{\mu\nu}$ are taken as operators of the Lorentz algebra, and the theory has Lorentz but no translation symmetry and, as such, exhibits violations of Poincaré symmetry.

Another theoretical setting to introduce the fundamental length is provided by a field theory effective model in usual commutative spacetime with manifest Lorentz invariance violation (LIV) [4]. In our case, the LIV is due to a deformation of the Lorentz dispersion relation of the field with the fundamental length taken as the deformation parameter. This is a simpler theoretical setting which has been extensively used for phenomenological and theoretical investigations, with the main purpose of explaining or finding new physics associated with Lorentz invariance violation. In this approach, a field theory effective model is usually formulated in a preferred class of frames, and is used to describe new physics in standard model phenomenology or astrophysical data which are identified by LIVs (see, e.g., [9] and the reviews [4]). In the Coleman and Glashow model [9], e.g. only invariance under rotations and translations is retained, which is also the case of the model that we study here. We consider a mass m scalar field with quartic self-interaction in usual commutative spacetime and with κ-deformed dispersion relation $(2\kappa)^2 \sinh[P^0/(2\kappa)]^2 - \mathbf{P}^2 = m^2$ [10, 11], in which the fundamental length q can be introduced by taking $\kappa = 1/(2q)$,

$$\left[\frac{1}{q}\sinh(qP^0)\right]^2 - \mathbf{P}^2 = m^2. \tag{1}$$

This deformed dispersion relation is obtained from the first Casimir invariant of the so-called κ-deformed Poincaré algebra in the standard basis [10, 11]. We consider such a model for the theoretical purpose of investigating the regularization effects coming from the sole κ-deformation of the Lorentz dispersion relation on the one-loop divergent diagrams, which can be an example of the regularizing effects of a fundamental length $q = 1/(2\kappa)$ in the context of the original Heisenberg proposal [1]. We will see in this simple model that the fundamental length provides a softening of ultraviolet divergences which shows the role of the κ-deformation as a possible regulator of a theory [11–13]. The simplicity of the model allows us to proceed with the calculations to obtain analytical expressions in terms of special functions and simple quadratures, thereby identifying the precise role of the fundamental length on the divergences of the one-loop diagrams. The simplicity of this kind of effective model with a κ-deformed dispersion relation made it possible to calculate the one-loop effective action for scalar field models under boundary conditions [14] and Casimir energies for an electromagnetic field model [15].

The fundamental length can also be introduced as the deformation parameter of deformations on both the Poincaré algebra and the algebra of spacetime coordinates. This is the case of the κ-deformed theory defined in the noncommutative κ-deformed Minkowski spacetime, and also obeying κ-deformed Poincaré algebra in the bicrossproduct basis [13, 16]. In this case, the noncommutative spacetime is obtained by duality with the κ-deformed Poincaré algebra in the bicrossproduct basis. The κ parameter does occur as a natural regularizing imaginary Pauli–Villars mass parameter in a κ-deformed scalar field theory with quartic self-interaction [17]. This kind of κ-deformed field theories presents a quite involved formalism to replace the Lorentz symmetry by a full κ-deformed symmetry of the theory [17–21] and also address the question of the LIV and its possible cure [22]. For those theories, it is essential to consider the co-algebra structure associated with the κ-deformed Poincaré algebra, in particular the deformed co-product composition of momenta [18, 19]. In the LIV formalism adopted here, usual composition of momenta and usual commutative spacetime are assumed with the κ-deformed dispersion relation, and the formalism is called an effective model to distinguish it from the above-mentioned κ-deformed field theories [17–21]. Let us note that the results of our simpler approach of an effective model with the LIV can suggest interesting properties to be investigated in such theories with the κ-deformed Poincaré algebra in κ-Minkowski spacetime.

We consider the effective model of a scalar field with the κ-deformed dispersion relation (1) and quartic self-interaction. Due to this deformation, the one-loop primitively divergent diagrams of the theory are finite in the four-dimensional limit of dimensionally regularized diagrams, because in this deformed case there is no pole at the physical dimension of spacetime [23]. This property motivates the investigation of the analytical structure of the diagrams for arbitrary dimensions, which we present here. The rest of the paper is organized as follows. In section 2, we give a brief account of the model with the finite four-dimensional limits of the primitively divergent diagrams of the theory following the original method of dimensional regularization of 't Hooft and Veltman [24]. This account sets the stage for the main result of this paper, the analytic structure of primitively divergent diagrams of the theory for arbitrary dimensions, which is presented in section 3. Finally, section 4 is left for the concluding remarks.

2. The scalar field model with a κ-deformed dispersion relation and quartic self-interaction

In order to define our model it is useful to set some notational conventions. First, we denote by ∂_q the q-differential operator $\partial_q = q^{-1}\sin(q\partial_0)$ introduced in [11]. Using this operator, we obtain from the κ-deformed dispersion relation (1) the κ-deformed Klein–Gordon operator $\partial_q^2 - \nabla^2 + m^2$. By introducing the convention that a bar over an index means that its range is $\{q, 1, 2, 3\}$, the κ-deformed Klein–Gordon operator is recasted into the simple form $\partial_{\bar{\mu}}\partial^{\bar{\mu}} + m^2$. The inverse of this operator with the prescription $m^2 \mapsto m^2 - \mathrm{i}\varepsilon$ is given in the Fourier representation by

$$\Delta(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^{\bar{\mu}}p_{\bar{\mu}} - m^2 + i\varepsilon}, \qquad (2)$$

where we have for notational convenience used the definition $p^{\bar{\mu}}p_{\bar{\mu}} = q^{-2}\sinh^2(qp^0) - \mathbf{p}^2$. A Lagrangian for the corresponding κ-deformed free scalar field is given by

$\mathcal{L}_0 = -(1/2)\phi\partial_{\bar{\mu}}\partial^{\bar{\mu}}\phi - (1/2)m^2\phi^2$ [11], to which we add a quartic self-interaction term with a coupling constant g to obtain

$$\mathcal{L} = -\frac{1}{2}\phi\partial_{\bar{\mu}}\partial^{\bar{\mu}}\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4. \tag{3}$$

The functional quantization of this model can be implemented along the same lines as the quantization of the usual non-deformed theory in order to obtain the two-point function

$$\Delta_c^{(2)}(x_1 - x_2) = \mathrm{i}\Delta(x_1 - x_2) - \frac{g}{2}\Delta(0)\int \mathrm{d}^4 z\, \Delta(x_1 - z)\Delta(z - x_2), \tag{4}$$

where $\Delta(0)$ is obtained from (2), and the four-point function is given by

$$\begin{aligned}\Delta_c^{(4)}(x_1, x_2, x_3, x_4) = &-\mathrm{i}g\int \mathrm{d}^4 z\, \Delta(x_1 - z)\Delta(x_2 - z)\Delta(x_3 - z)\Delta(x_4 - z)\\ &+ g^2\frac{1}{2}\int \mathrm{d}^4 z\, \mathrm{d}^4 z'\, \Delta(x_1 - z)\Delta(x_2 - z)[\Delta(z - z')]^2\Delta(z' - x_3)\Delta(z' - x_4)\\ &+ g^2\frac{1}{2}\int \mathrm{d}^4 z\, \mathrm{d}^4 z'\, \Delta(x_1 - z)\Delta(x_3 - z)[\Delta(z - z')]^2\Delta(z' - x_2)\Delta(z' - x_4)\\ &+ g^2\frac{1}{2}\int \mathrm{d}^4 z\, \mathrm{d}^4 z'\, \Delta(x_1 - z)\Delta(x_4 - z)[\Delta(z - z')]^2\Delta(z' - x_3)\Delta(z' - x_2).\end{aligned} \tag{5}$$

The two-point function (4) has the Fourier representation

$$\Delta_c^{(2)}(x_1 - x_2) = \mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x_1 - x_2)}}{p^{\bar{\mu}}p_{\bar{\mu}} - m^2 + \mathrm{i}\varepsilon}\left[1 + \frac{\Sigma_1(m^2)}{p^{\bar{\mu}}p_{\bar{\mu}} - m^2 + \mathrm{i}\varepsilon}\right], \tag{6}$$

where we have defined $\Sigma_1(m^2) = \mathrm{i}g\Delta(0)/2$ and used (2) to obtain

$$\Sigma_1(m^2) = \frac{\mathrm{i}}{2}g\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{p^{\bar{\mu}}p_{\bar{\mu}} - m^2 + \mathrm{i}\varepsilon}. \tag{7}$$

The Fourier representation of the four-point function (5) is given by

$$\begin{aligned}\Delta_c^{(4)}(x_1, x_2, x_3, x_4) = &\int \prod_{i=1}^{4}\left(\frac{[\mathrm{d}^4 p_i/(2\pi)^4]\,\mathrm{e}^{-\mathrm{i}p_i\cdot x_i}}{q^{-2}\sinh^2\left(qp_i^0\right) - \mathbf{p}_i^2 - m^2 + \mathrm{i}\varepsilon}\right)(2\pi)^4\delta^{(4)}(p_1 + p_2 + p_3 + p_4)\\ &\times\left[-\mathrm{i}g + \frac{1}{2}g^2\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{[(p - s)^{\bar{\mu}}(p - s)_{\bar{\mu}} - m^2 + \mathrm{i}\varepsilon](p^{\bar{\mu}}p_{\bar{\mu}} - m^2 + i\varepsilon)}\right.\\ &+ \frac{1}{2}g^2\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{[(p - t)^{\bar{\mu}}(p - t)_{\bar{\mu}} - m^2 + \mathrm{i}\varepsilon](p^{\bar{\mu}}p_{\bar{\mu}} - m^2 + \mathrm{i}\varepsilon)}\\ &\left.+ \frac{1}{2}g^2\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{[(p - u)^{\bar{\mu}}(p - u)_{\bar{\mu}} - m^2 + \mathrm{i}\varepsilon](p^{\bar{\mu}}p_{\bar{\mu}} - m^2 + \mathrm{i}\varepsilon)}\right],\end{aligned} \tag{8}$$

where we have used the definitions

$$(p - s)^{\bar{\mu}}(p - s)_{\bar{\mu}} = \frac{1}{q^2}\sinh^2[q(p^0 - s^0)] - (\mathbf{p} - \mathbf{s})^2, \tag{9}$$

and the analogous two definitions with t and u replacing s, being $s = p_1 + p_2$, $t = p_1 + p_3$ and $u = p_1 + p_4$ the Mandelstam variables.

By simple inspection, it is seen that those expressions for the Green's functions are divergent. For example, as in the non-deformed case, the quantity $\Sigma_1(m^2)$ in (7) is divergent. Therefore, κ-deformation does not render finite the Green's functions of the theory in perturbative calculations, and we are forced to resort to the usual process of regularization and renormalization in order to define those functions. We use here the dimensional regularization method [24, 25] following closely the original formalism of 't Hooft and Veltman [24]. In this regularization method, an ill-defined expression in four-dimensional spacetime, as the κ-deformed self-energy (7), is substituted by a well-defined expression in 2ω-dimensional spacetime, where ω is in some domain of the complex plane of spacetime dimensions. For the self-energy, we have e.g. the expression

$$\frac{2}{\mathrm{i}g}\Sigma_1(m^2,\omega) = (\mu^2)^{2-\omega}\int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}}\frac{1}{q^{-2}\sinh^2(qp^0) - \mathbf{p}^2 - m^2 + \mathrm{i}\varepsilon} \tag{10}$$

or, more explicitly, the expression

$$\frac{2}{\mathrm{i}g}\Sigma_1(m^2,\omega) = \frac{(4\pi\mu^2)^{2-\omega}}{16\pi^4}\frac{\pi^{3/2}}{\Gamma(\omega-1/2)} \times \int_0^\infty \mathrm{d}r^2 (r^2)^{\omega-3/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{q^{-2}\sinh^2(qp^0) - r^2 - m^2 + \mathrm{i}\varepsilon}, \tag{11}$$

which is certainly well defined for ω in the domain $1/2 < \mathrm{Re}\,\omega < 1$, since $q^{-2}\sinh^2(qp^0) \geqslant (p^0)^2$ for any real p^0. In this well-defined expression, we can make the change of the integration variable $q^{-1}\sinh(qp^0) \mapsto p^0$ to obtain

$$\frac{2}{\mathrm{i}g}\Sigma_1(m^2,\omega) = \frac{(4\pi\mu^2)^{2-\omega}}{16\pi^4}\frac{\pi^{3/2}}{\Gamma(\omega-1/2)} \times \int_0^\infty \mathrm{d}r^2 (r^2)^{\omega-3/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{\sqrt{1+q^2p_0^2}}\frac{1}{(p^0)^2 - r^2 - m^2 + i\varepsilon}, \tag{12}$$

where the factor $1\big/\sqrt{1+q^2p_0^2}$ in the integrand behaves asymptotically as $1/|p^0|$, and obeys the inequality

$$\frac{1}{\sqrt{1+q^2p_0^2}} \leqslant 2\kappa\frac{|p_0| + 2\kappa}{p^2 + (2\kappa)^2}, \tag{13}$$

which shows that the factor $1\big/\sqrt{1+q^2p_0^2}$ is majorized by a fermion-like propagator, making it safe to get from this factor a -1 contribution to the power-counting. After the change of the integration variable we make a trivial analytic continuation of the self-energy (12) to the strip $1/2 < \mathrm{Re}\,\omega < 3/2$ in the complex ω-plane. The next step in the dimensional regularization method is to extend this expression for the self energy to a neighborhood of $\omega = 2$ by partial integrations. A single partial integration leads us to

$$\frac{2}{\mathrm{i}g}\Sigma_1(m^2,\omega) = \frac{(4\pi\mu^2)^{2-\omega}}{16\pi^4}\frac{\pi^{3/2}}{\Gamma(\omega-1/2)}\frac{1}{\omega-3/2}\left[m^2\int_{-\infty}^{\infty}\mathrm{d}p_0\int_0^\infty \mathrm{d}r^2(r^2)^{\omega-3/2} \times \frac{1}{(p^2-m^2+i\varepsilon)^2\left(1+q^2p_0^2\right)^{1/2}} - \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}p_0\int_0^\infty \mathrm{d}r^2\frac{(r^2)^{\omega-3/2}}{(p^2-m^2+i\varepsilon)\left(1+q^2p_0^2\right)^{3/2}}\right], \tag{14}$$

which is well defined in the strip $1/2 < \mathrm{Re}\,\omega < 5/2$ punctured at pole $\omega = 3/2$. In the non-deformed case, we would obtain an expression with a pole at physical dimension $2\omega = 4$, and a subtraction of the term containing this pole would be necessary before taking the limit of physical dimension. Here, however, this usual 'subtraction of infinities'of the quantum field theory is not necessary, since (14) has no pole at physical dimension. Indeed, in this κ-deformed case, the pole in the domain $1/2 < \mathrm{Re}\,\omega < 5/2$ appears for $2\omega = 3$. Consequently, we can take $\omega = 2$ in the self-energy (14) to obtain a finite result at the physical dimension $2\omega = 4$ of spacetime. Although the finiteness of self-energy at physical dimension is settled on by this result, it is convenient to make a further analytical continuation by partial integration to arrive at

$$\frac{2}{\mathrm{i}g}\Sigma_1(m^2,\omega) = \frac{(4\pi\mu^2)^{2-\omega}}{16\pi^4}\frac{\pi^{3/2}}{\Gamma(\omega-1/2)}\frac{1}{(\omega-3/2)\,(\omega-5/2)}\left[2m^4\int_{-\infty}^{\infty}\mathrm{d}p_0\int_0^{\infty}\mathrm{d}r^2\right.$$
$$\times\frac{(r^2)^{\omega-3/2}}{(p^2-m^2+\mathrm{i}\varepsilon)^3\left(1+q^2p_0^2\right)^{1/2}} - m^2\int_{-\infty}^{\infty}\mathrm{d}p_0\int_0^{\infty}\mathrm{d}r^2\frac{(r^2)^{\omega-3/2}}{(p^2-m^2+\mathrm{i}\varepsilon)^2\left(1+q^2p_0^2\right)^{3/2}}$$
$$\left.+\frac{3}{4}\int_{-\infty}^{\infty}\mathrm{d}p_0\int_0^{\infty}\mathrm{d}r^2\frac{(r^2)^{\omega-3/2}}{(p^2-m^2+\mathrm{i}\varepsilon)\left(1+q^2p_0^2\right)^{5/2}}\right], \tag{15}$$

which is well defined in the strip $1/2 < \mathrm{Re}\,\omega < 7/2$ punctured at the poles $\omega = 3/2$ and $\omega = 5/2$. From this expression, the non-deformed limit is obtained in its dimensional regularized form [26]. This expression also shows that self-energy in this punctured strip is finite at dimension $2\omega = 6$, and has its poles at dimensions $2\omega = 3$ and $2\omega = 5$. These poles are shifted in relation to the two first poles of the non-deformed self-energy of $1/2$ to the right in the complex ω-plane, since in the non-deformed case the poles occur for dimension $2\omega = 2, 4, 6, \ldots$ [26]. We may ask if all the poles for the non-deformed self-energy are given by this shifting of the poles in the non-deformed case, i.e. if in the deformed case the poles occur at $\omega = 3/2, 5/2, \ldots$ points in the complex ω-plane. To answer this, we must make a full analytical continuation of the deformed self-energy (15), a problem to be considered in the following section. Now, we take the limit of physical dimension $2\omega \to 4$ in (15) to obtain the following finite value for the self-energy in the deformed case:

$$\frac{2}{\mathrm{i}g}\Sigma_1(m^2) = -\frac{1}{2\pi^4}\left[2m^4\int_{-\infty}^{\infty}\mathrm{d}p_0\int_0^{\infty}\mathrm{d}r^2\frac{r}{(p^2-m^2+\mathrm{i}\varepsilon)^3\left(1+q^2p_0^2\right)^{1/2}}\right.$$
$$-m^2\int_{-\infty}^{\infty}\mathrm{d}p_0\int_0^{\infty}\mathrm{d}r^2\frac{r}{(p^2-m^2+\mathrm{i}\varepsilon)^2\left(1+q^2p_0^2\right)^{3/2}}$$
$$\left.+\frac{3}{4}\int_{-\infty}^{\infty}\mathrm{d}p_0\int_0^{\infty}\mathrm{d}r^2\frac{r}{(p^2-m^2+\mathrm{i}\varepsilon)\left(1+q^2p_0^2\right)^{5/2}}\right]. \tag{16}$$

Now, let us consider the four-point function (8), which is not well defined due to its dependence on the three divergent vertex integrals $\Gamma^{(4)}(m,s,\omega)$, $\Gamma^{(4)}(m,t,\omega)$ and $\Gamma^{(4)}(m,u,\omega)$ defined by

$$\frac{2}{g^2}\Gamma^{(4)}(m,s,\omega) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{[(p-s)^{\bar\mu}(p-s)_{\bar\mu}-m^2+\mathrm{i}\varepsilon](p^{\bar\mu}p_{\bar\mu}-m^2+\mathrm{i}\varepsilon)}, \tag{17}$$

and the expressions obtained by replacing s by t or u. The same method of dimensional regularization applied to expression (17) now leads us to

$$\frac{2}{g^2}\Gamma^{(4)}(m,s,\omega) = (\mu^2)^{2-\omega}\int\frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}}\frac{1}{\sqrt{1+q^2p_0^2}}\frac{1}{p^2-m^2+\mathrm{i}\varepsilon}$$
$$\times\frac{1}{p_0^2-q^{-1}\sinh(2qs^0)p_0\sqrt{1+q^2p_0^2}+\left(1+2q^2p_0^2\right)q^{-2}\sinh^2(qs^0)-(\mathbf{p}-\mathbf{s})^2-m^2+\mathrm{i}\varepsilon}, \tag{18}$$

which is well defined in the strip $1/2 < \mathrm{Re}\,\omega < 5/2$. Since this function is regular at $\omega = 2$, we immediately obtain for the physical dimension the finite quantity

$$\frac{2}{g^2}\Gamma^{(4)}(m,s,2) = \int \frac{d^4 p}{(2\pi)^4}\frac{1}{\sqrt{1+q^2 p_0^2}}\frac{1}{p^2 - m^2 + i\varepsilon} \times \frac{1}{p_0^2 - q^{-1}\sinh(2qs^0)p_0\sqrt{1+q^2p_0^2} + \left(1+2q^2p_0^2\right)q^{-2}\sinh^2(qs^0) - (\mathbf{p}-\mathbf{s})^2 - m^2 + i\varepsilon}. \tag{19}$$

Therefore, the vertex function is also finite at physical dimension due to the κ-deformation. From our previous experience with the self-energy, we expect that the pole at $\omega = 2$ of the non-deformed function $\Gamma^{(4)}(m,s,\omega)|_{q=0}$ has been shifted to the boundary point $\omega = 5/2$ of the strip $1/2 < \mathrm{Re}\,\omega < 5/2$. At this point, we may also ask if the real poles of the vertex function in the deformed case are also obtained from the shifting of $1/2$ to the right of the poles in the non-deformed case, i.e. if in the deformed case the poles occur at $\omega = 5/2, 7/2, \ldots$ points in the complex ω-plane. For this, we need to find the appropriate analytical continuation of the vertex function, a problem that we consider in the following section.

3. Analytic structure of the self-energy and vertex diagrams

In this section, we obtain analytical extensions of the deformed self-energy and vertex functions in the complex ω-plane in order to compare their poles at real dimension with the known poles of the non-deformed corresponding functions. We start with the self-energy for which we have the function (12) with domain $1/2 < \mathrm{Re}\,\omega < 3/2$. We have shown that this function can be analytically continued to the neighborhood of $\omega = 2$, where we reach physical dimension with a finite value for deformed the self-energy. Now let us proceed to the analytical continuation of this function to disclose its pole structure in the whole real ω-axis. First of all, the integral in p^0 which appears in (12) can be solved by an elementary application of Cauchy's theorem in the complex p^0-plane. The integration path can be closed in the upper half-plane in such a way that the branch cut $[iq, i\infty)$ is bypassed and the pole at $p^0 = -(r^2+m^2)^{1/2} + i\varepsilon$ is enclosed. This gives us

$$\int_{-\infty}^{\infty}\frac{dp^0}{\sqrt{1+q^2p_0^2}}\frac{1}{p^2-m^2+i\varepsilon} = -\frac{i\pi + \cosh^{-1}[1+2q^2(r^2+m^2)]}{(r^2+m^2)^{1/2}\sqrt{1+q^2(r^2+m^2)}}. \tag{20}$$

By substituting this result into (12), we obtain after some simplifications

$$\frac{2}{ig}\Sigma_1(m^2,\omega) = -\frac{(4\pi\mu^2)^{2-\omega}}{16\pi^4}\frac{\pi^{3/2}}{\Gamma(\omega-1/2)}\left[i\pi\int_0^\infty dr^2\frac{(r^2)^{\omega-3/2}}{(r^2+m^2)^{1/2}\sqrt{1+q^2(r^2+m^2)}} + \int_0^\infty dr^2 (r^2)^{\omega-3/2}\frac{\cosh^{-1}[1+2q^2(r^2+m^2)]}{(r^2+m^2)^{1/2}\sqrt{1+q^2(r^2+m^2)}}\right]. \tag{21}$$

The first integral in this expression is given in terms of beta and hypergeometric functions by ([27], 3.197 (1))

$$\int_0^\infty dr^2 (r^2)^{\omega-3/2}(r^2+m^2)^{-1/2}[1+q^2(r^2+m^2)]^{-1/2} = \frac{(m^2)^{\omega-1}}{(1+q^2m^2)^{1/2}} \times \Gamma\left(\omega-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-\omega\right)F\left(\frac{1}{2},\omega-\frac{1}{2},1,\frac{1}{1+q^2m^2}\right). \tag{22}$$

The second integral in (21) has the expected zero value at the limit $q = 0$ for dimensions 2ω such that $1/2 < \mathrm{Re}\,\omega < 3/2$. Since we are now interested in the localization of the poles in the complex ω-plane for the deformed theory, we will consider $q > 0$ and change of the integration variable in the second integral in (21) to

$$v = \cosh^{-1}[1 + 2q^2(r^2 + m^2)]. \tag{23}$$

With this change of the variable (which renders the limit $q = 0$ a delicate matter as discussed bellow), we obtain for the second integral an expression in terms of a derivative of a Legendre function in relation to its order, and this derivative can be expressed in terms of gamma functions ([27], 8.715 (2)). In this way, we obtain for the second integral in (21)

$$\int_0^\infty \mathrm{d}r^2 (r^2)^{\omega-3/2} \frac{\cosh^{-1}[1 + 2q^2(r^2 + m^2)]}{(r^2 + m^2)^{1/2}\sqrt{1 + q^2(r^2 + m^2)}} = \frac{(m^2)^{\omega-1}}{(1 + q^2m^2)^{1/2}} \times \Gamma\left(\omega - \frac{1}{2}\right)\Gamma\left(\frac{3}{2} - \omega\right)[(qm)^{-1}(1 + q^2m^2)^{1/2}]^{\omega-1/2}. \tag{24}$$

Now we take both integrals (22) and (24) into (21) to obtain for the self-energy the expression

$$\frac{2}{\mathrm{i}g}\Sigma_1(m^2, \omega) = -\frac{m^2}{16\pi^4}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega}\pi^{3/2}\Gamma\left(\frac{3}{2} - \omega\right)\left[\frac{\mathrm{i}\pi}{(1 + q^2m^2)^{1/2}} \times F\left(\frac{1}{2}, \omega - \frac{1}{2}, 1, \frac{1}{1 + q^2m^2}\right) + (qm)^{1/2-\omega}(1 + q^2m^2)^{\omega/2-3/4}\right], \tag{25}$$

where $1/2 < \mathrm{Re}\,\omega < 3/2$. We should note that this is the domain of validity of the original expression (12) and also the domain on which the calculations resulting in (25) can be justified. However, now it is trivial to make the analytical continuation of the right-hand side of (25) to the whole complex ω-plane, except for simple poles at $\omega = 3/2, 5/2, \ldots$. In this way, we have obtained the structure of poles suggested by the calculations in the previous section. Therefore, we may say that in a κ-deformed theory, the self-energy is divergent only in odd-dimensional spacetimes and, in particular, it is convergent in even-dimensional spacetimes $2\omega = 2, 4, \ldots$. This property is in contrast with properties of the dimensional regularized self-energy in the non-deformed case, which diverges for spacetimes of even dimension. We should note that, although expression (25) for the self-energy displays the pole structure in the complex ω-plane of a deformed theory ($q > 0$), it is not clear that it can provide the correct limit of non-deformation ($q \to 0$), since it was obtained from the second integral in (21) by the change of the integration variable $v = \cosh^{-1}[1+2q^2(r^2+m^2)]$, which is badly ill defined for $q = 0$. The expected correct limit for non-deformation should be obtained making $q \to 0$ in expression (21), prior to the change of the integration variable. Therefore, it is interesting to note that a maneuvering of (25) in the complex ω-plane can provide us with the correct non-deformed limit. Indeed, after the analytical extension of (25) to the complex ω-plane punctured at $\omega = 3/2, 5/2, \ldots$, we proceed to its analytical restriction to a domain with $\mathrm{Re}\,\omega < 1/2$, say to the strip $0 < \mathrm{Re}\,\omega < 1/2$, in which we obtain the following non-deformed limit of (25):

$$\lim_{q\to 0}\Sigma_1(m^2, \omega) = -\frac{\mathrm{i}gm^2}{32\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega}\Gamma(1 - \omega), \tag{26}$$

which can be trivially analytically continued to the whole complex ω-plane, except for simple poles at $\omega = 1, 2, \ldots$. This is the usual non-deformed expression for the self-energy, which diverges for even dimensions $2\omega = 2, 4, \ldots$, in particular for the physical dimension into consideration, $2\omega = 4$. Now, back to (25) defined in the complex ω-plane punctured at

$\omega = 3/2, 5/2, \ldots$, we take the limit $2\omega \to 4$ to obtain the following finite result at physical dimension for the deformed theory:

$$\Sigma_1(m^2) = -\frac{gm^2}{8\pi^2}\frac{(1+q^2m^2)^{1/2}}{(qm)^2}\mathrm{E}\left(\frac{1}{1+q^2m^2}\right) + \mathrm{i}\frac{gm^2}{16\pi^2}\frac{(1+q^2m^2)^{1/4}}{(qm)^{3/2}}, \tag{27}$$

where E is the complete elliptic integral of the second kind. Since we are considering q as a natural parameter of the theory, we can say that the finiteness of the self-energy (27) is a consequence of the non-zero value of q, i.e. of the deformation. If we were considering q as an usual regularization parameter, we would have to eliminate it at the end of the calculations. In this case, taking the limit $q \to 0$, we get a divergent self-energy in (27) with quadratic maximal divergence (in the first term we have $\mathrm{E}(1) = 1$).

Let us now turn to the dimensionally regularized vertex function (18) with domain $1/2 < \mathrm{Re}\,\omega < 5/2$ in order to obtain its complete pole structure in the complex ω-plane. For the present purpose, it is convenient to change back to the original integration variable by means of the change of the variable $p^0 \mapsto q^{-1}\sinh(qp^0)$. We obtain the function

$$\frac{2}{g^2}\Gamma^{(4)}(m^2,s,\omega) = (\mu^2)^{2-\omega}\int\frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}}\frac{1}{q^{-2}\sinh^2(qp^0-qs^0)-(\mathbf{p}-\mathbf{s})^2-m^2+\mathrm{i}\varepsilon}$$
$$\times\frac{1}{p^{\bar\mu}p_{\bar\mu}-m^2+\mathrm{i}\varepsilon}, \tag{28}$$

with the domain $1/2 < \mathrm{Re}\,\omega < 5/2$, which can be written as

$$\Gamma^{(4)}(m^2,s,\omega) = \frac{1}{2}g^2(\mu^2)^{2-\omega}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}}$$
$$\times\frac{1}{[q^{-2}\sinh^2(qp^0)-q^{-2}\sinh(2qp^0-qs^0)\sinh(qs^0)z-(\mathbf{p}-\mathbf{s}z)^2-m^2+\mathrm{i}\varepsilon-\mathbf{s}^2z(1-z)]^2}. \tag{29}$$

where the Feynman representation for the product of propagators has been introduced. By making the usual change of the integration variable, $\mathbf{p} \mapsto \mathbf{p}' + \mathbf{s}z$, we obtain

$$\Gamma^{(4)}(m^2,s,\omega) = \frac{1}{2}g^2(\mu^2)^{2-\omega}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}}$$
$$\times\frac{1}{[q^{-2}\sinh^2(qp^0)-q^{-2}\sinh(2qp^0-qs^0)\sinh(qs^0)z-\mathbf{p}^2-m^2-\mathbf{s}^2z(1-z)+\mathrm{i}\varepsilon]^2}, \tag{30}$$

which can be recasted, after the integration on the solid angle, into the form

$$\Gamma^{(4)}(m^2,s,\omega) = g^2\frac{(4\pi\mu^2)^{2-\omega}}{32\pi^4}\frac{\pi^{3/2}}{\Gamma(\omega-1/2)}\int_0^1\mathrm{d}z\int_0^\infty\mathrm{d}r^2(r^2)^{\omega-3/2}\int_{-\infty}^\infty\mathrm{d}p^0$$
$$\times\frac{1}{[q^{-2}\sinh^2(qp^0)-q^{-2}\sinh(2qp^0-qs^0)\sinh(qs^0)z-r^2-m^2-\mathbf{s}^2z(1-z)+\mathrm{i}\varepsilon]^2}. \tag{31}$$

Now we restrict the domain of this function of ω to the strip $1/2 < \mathrm{Re}\,\omega < 3/2$ in order to write the integrand in it as a derivative in relation to m^2,

$$\Gamma^{(4)}(m^2,s,\omega) = g^2\frac{(4\pi\mu^2)^{2-\omega}}{32\pi^4}\frac{\pi^{3/2}}{\Gamma(\omega-1/2)}\frac{\partial}{\partial m^2}\int_0^1\mathrm{d}z\int_0^\infty\mathrm{d}r^2(r^2)^{\omega-3/2}\int_{-\infty}^\infty\mathrm{d}p^0$$
$$\times\frac{1}{q^{-2}\sinh^2(qp^0)-q^{-2}\sinh(2qp^0-qs^0)\sinh(qs^0)z-r^2-m^2-\mathbf{s}^2z(1-z)+\mathrm{i}\varepsilon}. \tag{32}$$

After some elementary manipulations, which includes the change of the integration variable $\zeta = \mathrm{e}^{2qp^0}$, this equation is brought into the form

$$\Gamma^{(4)}(m^2, s, \omega) = g^2 \frac{(4\pi\mu^2)^{2-\omega}}{32\pi^4} \frac{\pi^{3/2}}{\Gamma(\omega - 1/2)} \frac{\partial}{\partial m^2} \int_0^1 \mathrm{d}z \int_0^\infty \mathrm{d}r^2 (r^2)^{\omega-3/2} \times 2q \int_0^\infty \frac{\mathrm{d}\zeta}{a\zeta^2 - 2(b - \mathrm{i}2q^2\varepsilon)\zeta + c}, \tag{33}$$

where we have introduced the positive parameters

$$a = 1 - 2\mathrm{e}^{-qs^0}\sinh(qs^0)z, \qquad c = 1 + 2\mathrm{e}^{qs^0}\sinh(qs^0)z,$$
$$b = 1 + 2q^2[r^2 + m^2 + \mathbf{s}^2 z(1-z)], \tag{34}$$

with the condition

$$(b^2 - ac)/(2q)^2 = m^2 + r^2 - s_{\bar{\mu}}s^{\bar{\mu}}z(1-z) + q^2[m^2 + r^2 + \mathbf{s}^2 z(1-z)]^2 > 0, \tag{35}$$

for any values of the integration variables r^2 and z. This condition is obviously satisfied in the Euclidean region ($-s_{\bar{\mu}}s^{\bar{\mu}} \gg m^2$), to which we restrict the s parameter from now on. In the non-deformed limit, $b^2 - ac > 0$ reduces to the inequality $m^2 + r^2 - s_\mu s^\mu z(1-z) > 0$ which gives rise to the condition that keeps real values of $s_\mu s^\mu$ out of the physical cut in the complex $s_\mu s^\mu$-plane, namely, $s_\mu s^\mu < 4m^2$. In the present deformed case, a precise localization of the branch point requires the solution of transcendental equations and is not necessary for our purposes. At any rate, we can assume condition (35) and solve the integral in ζ which appears in (33) by an elementary application of Cauchy's theorem in the complex ζ-plane. The integration path is obtained by closing the path $(\mathrm{i}\infty, \mathrm{i}0] \cup [0, \infty)$ in the first quadrant. In this way (33) acquires the form

$$\Gamma^{(4)}(m^2, s, \omega) = -g^2 \frac{(4\pi\mu^2)^{2-\omega}}{32\pi^4} \frac{\pi^{3/2}}{\Gamma(\omega - 1/2)} \frac{\partial}{\partial m^2} \int_0^1 \mathrm{d}z \int_0^\infty \mathrm{d}r^2 (r^2)^{\omega-3/2} \times \frac{1}{(r^2 + m^2 - \mathrm{i}\varepsilon - s^{\bar{\mu}}s_{\bar{\mu}}z(1-z) + q^2[r^2 + m^2 + \mathbf{s}^2 z(1-z)]^2)^{1/2}} \times \left[\mathrm{i}\pi + \cosh^{-1}\left(\frac{1 + 2q^2[r^2 + m^2 + \mathbf{s}^2 z(1-z)]}{\sqrt{1 + 4\sinh^2(qs^0)z(1-z)}}\right)\right]. \tag{36}$$

By performing in this expression the derivative in relation to m^2, we obtain the expression

$$\Gamma^{(4)}(m^2, s, \omega) = g^2 \frac{(4\pi\mu^2)^{2-\omega}}{64\pi^4} \frac{\pi^{3/2}}{\Gamma(\omega - 1/2)} \int_0^1 \mathrm{d}z \times \left\{ \mathrm{i}\pi(1 + 2q^2M^2) \int_0^\infty \mathrm{d}r^2 \frac{(r^2)^{\omega-3/2}}{\left[r^2 + M_q^2 - \mathrm{i}\varepsilon + q^2(r^2 + M^2)^2\right]^{3/2}} + \mathrm{i}\pi 2q^2 \int_0^\infty \mathrm{d}r^2 \frac{(r^2)^{\omega-1/2}}{\left[r^2 + M_q^2 - \mathrm{i}\varepsilon + q^2(r^2 + M^2)^2\right]^{3/2}} - 2q \int_0^\infty \mathrm{d}r^2 \frac{(r^2)^{\omega-3/2}}{r^2 + M_q^2 - \mathrm{i}\varepsilon + q^2(r^2 + M^2)^2} + \int_0^\infty \mathrm{d}r^2 \frac{(r^2)^{\omega-3/2}[1 + 2q^2(r^2 + M^2)]}{\left[r^2 + M_q^2 - \mathrm{i}\varepsilon + q^2(r^2 + M^2)^2\right]^{3/2}} \times \cosh^{-1}\left[\frac{1 + 2q^2(r^2 + M^2)}{\sqrt{1 + 4\sinh^2(qs^0)z(1-z)}}\right] \right\}, \tag{37}$$

where

$$M_q = \sqrt{m^2 - s^{\bar{\mu}}s_{\bar{\mu}}z(1-z)}, \qquad M = \sqrt{m^2 + \mathbf{s}^2 z(1-z)}. \tag{38}$$

Now, we are left with the r^2 integrations in (37). The three first integrals are given in terms of gamma functions and hypergeometric functions that depend on ω and the latter also on the parameter

$$\eta_q = \frac{1 + 2q^2(m^2 + \mathbf{s}^2 z(1-z)) - \mathrm{i}\varepsilon}{2q\sqrt{m^2 - s_{\bar{\mu}} s^{\bar{\mu}} z(1-z) + q^2[m^2 + \mathbf{s}^2 z(1-z)]^2}}, \tag{39}$$

which is well defined, thanks to condition (35) (for $r^2 = 0$). The non-zero imaginary part of η in (39) allows us to write ([27], 3.252 (11))

$$\int_0^\infty \mathrm{d}r^2 \frac{(r^2)^{\omega-3/2}}{\left[r^2 + M_q^2 - \mathrm{i}\varepsilon + q^2(r^2+M^2)^2\right]^{3/2}} = 2q^{3/2-\omega} \frac{\left(M_q^2 + q^2 M^4\right)^{\omega/2-5/4}}{1 + 2q^2M^2 + 2q\left(M_q^2 + q^2M^4\right)^{1/2}}$$
$$\times \Gamma\left(\omega - \frac{1}{2}\right)\Gamma\left(\frac{7}{2} - \omega\right) F\left(\omega - \frac{3}{2}, \frac{5}{2} - \omega; 2, \frac{1-\eta_q}{2}\right), \tag{40}$$

$$\int_0^\infty \mathrm{d}r^2 \frac{(r^2)^{\omega-1/2}}{\left[r^2 + M_q^2 - \mathrm{i}\varepsilon + q^2(r^2+M^2)^2\right]^{3/2}} = 2q^{1/2-\omega} \frac{\left(M_q^2 + q^2 M^4\right)^{\omega/2-3/4}}{1 + 2q^2M^2 + 2q\left(M_q^2 + q^2M^4\right)^{1/2}}$$
$$\times \Gamma\left(\omega + \frac{1}{2}\right)\Gamma\left(\frac{5}{2} - \omega\right) F\left(\omega - \frac{1}{2}, \frac{3}{2} - \omega; 2, \frac{1-\eta_q}{2}\right), \tag{41}$$

and

$$\int_0^\infty \mathrm{d}r^2 \frac{(r^2)^{\omega-3/2}}{r^2 + M_q^2 - \mathrm{i}\varepsilon + q^2(r^2+M^2)^2} = 2q^{1-\omega} \frac{\left(M_q^2 + q^2 M^4\right)^{\omega/2-1}}{\left[1 + 2q^2M^2 + 2q\left(M_q^2 + q^2M^4\right)^{1/2}\right]^{1/2}}$$
$$\times \Gamma\left(\omega - \frac{1}{2}\right)\Gamma\left(\frac{5}{2} - \omega\right) F\left(\omega - 1, 2 - \omega; \frac{3}{2}, \frac{1-\eta_q}{2}\right). \tag{42}$$

which are valid for $1/2 < \mathrm{Re}\,\omega < 7/2, -1/2 < \mathrm{Re}\,\omega < 5/2$ and $1/2 < \mathrm{Re}\,\omega < 5/2$, respectively. Therefore, those three equations are simultaneously valid in the domain $1/2 < \mathrm{Re}\,\omega < 3/2$ in which we are considering the vertex function (37). The presence of the inverse hyperbolic cosinus in the last r^2 integral in (37) will require a procedure similar to the one used in the last integral of the self-energy (21), only more involved and trickier. In the last integral in (37), we change the integration variable to

$$w = \cosh^{-1}\left[\frac{1 + 2q^2(r^2+M^2)}{\sqrt{1 + 4\sinh^2(qs^0)z(1-z)}}\right] (q > 0) \tag{43}$$

in order to obtain

$$\int_0^\infty \mathrm{d}r^2 (r^2)^{\omega-3/2} \frac{1 + 2q^2(r^2+M^2)}{\left[r^2 + M_q^2 + q^2(r^2+M^2)^2\right]^{3/2}} \cosh^{-1}\left[\frac{1 + 2q^2(r^2+M^2)}{\sqrt{1 + 4\sinh^2(qs^0)z(1-z)}}\right]$$
$$= \frac{2^{3/2}(2q^2)^{2-\omega}}{[1 + 4\sinh^2(qs^0)z(1-z)]^{5/4-\omega/2}}$$
$$\times \int_{w_0}^\infty \mathrm{d}w\, w \frac{\cosh(w)}{\sinh^2(w)}[\cosh(w) - \cosh(w_0)]^{\omega-3/2}, \tag{44}$$

where $w_0 = \cosh^{-1}[(1 + 2q^2M^2)(1 + 4\sinh^2(qs^0)z(1-z))^{-1/2}]$. An integration by parts on the right-hand side of (44) transforms it into integrals that can be expressed in terms of gamma and hyperbolic functions

$$\int_0^\infty dr^2 (r^2)^{\omega-3/2} \frac{1+2q^2(r^2+M^2)}{\left[r^2+M_q^2-i\varepsilon+q^2(r^2+M^2)^2\right]^{3/2}} \cosh^{-1}\left[\frac{1+2q^2(r^2+M^2)}{\sqrt{1+4\sinh^2(qs^0)z(1-z)}}\right]$$
$$= 2(q^2)^{2-\omega}\left(q^2M_q^2+q^4M^4\right)^{\omega/2-5/4}\Gamma\left(\omega-\frac{1}{2}\right)\Gamma\left(\frac{5}{2}-\omega\right)$$
$$+ 2^{3/2-\omega}\Gamma\left(\frac{3}{2}-\omega\right)\left(1+2q^2M^2+\sqrt{1+4\sinh^2(qs^0)z(1-z)}\right)^{\omega-3/2}$$
$$\times \frac{(q^2)^{2-\omega}}{\sqrt{1+4\sinh^2(qs^0)z(1-z)}}\left[1-\left(\frac{1+2q^2M^2-\sqrt{1+4\sinh^2(qs^0)z(1-z)}}{1+2q^2M^2+\sqrt{1+4\sinh^2(qs^0)z(1-z)}}\right)^{\omega-3/2}\right], \tag{45}$$

for $1/2 < \mathrm{Re}\,\omega < 3/2$. In this way, all the four results (40)–(42) and (45) are valid in the common domain $1/2 < \mathrm{Re}\,\omega < 3/2$. By substituting these results into (37), we obtain in this domain

$$\Gamma^{(4)}(m^2,s,\omega) = ig^2\frac{(4\pi\mu^2)^{2-\omega}}{32\pi^4}\pi^{5/2}\int_0^1 dz \frac{dz}{1+2q^2M^2+2q\left(M_q^2+q^2M^4\right)^{1/2}}$$
$$\times\left(M_q^2+q^2M^4\right)^{\omega/2-5/4}\left\{(1+2q^2M^2)q^{3/2-\omega}\Gamma\left(\frac{7}{2}-\omega\right)F\left(\omega-\frac{3}{2},\frac{5}{2}-\omega;2,\frac{1-\eta_q}{2}\right)\right.$$
$$\left.+2q^{5/2-\omega}\left(M_q^2+q^2M^4\right)^{1/2}\frac{\Gamma(\omega+1/2)}{\Gamma(\omega-1/2)}\Gamma\left(\frac{5}{2}-\omega\right)F\left(\omega-\frac{1}{2},\frac{3}{2}-\omega;2,\frac{1-\eta_q}{2}\right)\right\}$$
$$+g^2\frac{(4\pi\mu)^{2-\omega}}{32\pi^4}\pi^{3/2}\int_0^1 dz\left\{-2q^{2-\omega}\frac{\left(M_q^2+q^2M^4\right)^{\omega/2-1}}{\left[1+2q^2M^2+2q\left(M_q^2+q^2M^4\right)^{1/2}\right]^{1/2}}\right.$$
$$\times\Gamma\left(\frac{5}{2}-\omega\right)F\left(\omega-1,2-\omega;\frac{3}{2},\frac{1-\eta_q}{2}\right)+(q^2)^{2-\omega}\left(q^2M_q^2+q^4M^4\right)^{\omega/2-5/4}$$
$$\times\Gamma\left(\frac{5}{2}-\omega\right)+2^{3/2-\omega}\Gamma\left(\frac{3}{2}-\omega\right)\left(1+2q^2M^2+\sqrt{1+4\sinh^2(qs^0)z(1-z)}\right)^{\omega-3/2}$$
$$\left.\times\frac{(q^2)^{2-\omega}}{\sqrt{1+4\sinh^2(qs^0)z(1-z)}}\left[1-\left(\frac{1+2q^2M^2-\sqrt{1+4\sinh^2(qs^0)z(1-z)}}{1+2q^2M^2+\sqrt{1+4\sinh^2(qs^0)z(1-z)}}\right)^{\omega-3/2}\right]\right\}. \tag{46}$$

where M_q, M and η_q are given in (38) and (39), and condition (35) is enforced. Now, we make a trivial analytical continuation of the vertex function to the complex ω-plane with exception of the simple poles at $\omega = 5/2, 7/2\ldots$. Therefore, we have also obtained for the vertex function (46) the result that κ-deformation shift the poles from even-dimensional to odd-dimensional spacetime. To display this pole structure, we had to assume in the change of variable (43) the condition $q > 0$, which prevent us of expecting from (46) the correct non-deformed limit when $q \to 0$. We have obtained the correct form of the self-energy in this limit by performing first a simple restriction of its domain to a region in which $\mathrm{Re}\,\omega < 1/2$. Now, for the vertex function (46), we will first need to use transformation formulas for the hypergeometric functions ([27], 9.132 (1)) in order to obtain

$$\Gamma^{(4)}(m^2, s, \omega) = \mathrm{i}\frac{g^2}{32\pi^2}\int_0^1 \mathrm{d}z \left(\frac{4\pi\mu^2}{M_q^2 + q^2M^4}\right)^{2-\omega} \frac{1}{\left[1 + 2q^2M^2 + 2q\left(M_q^2 + q^2M^4\right)^{1/2}\right]^{\omega-1/2}}$$
$$\times \left\{(1 + 2q^2M^2)\left[\Gamma(2-\omega)F\left(\omega - \frac{3}{2}, \omega - \frac{1}{2}, 2\omega - 3, \frac{2}{1+\eta_q}\right)\right.\right.$$
$$\left. + \left(\frac{1}{2+2\eta_q}\right)^{4-2\omega}\Gamma(\omega-2)\frac{\Gamma(7/2-\omega)}{\Gamma(\omega-1/2)}F\left(\frac{5}{2} - \omega, \frac{7}{2} - \omega, 5 - 2\omega, \frac{2}{1+\eta_q}\right)\right]$$
$$+ \frac{(2\omega-1)q^2\left(M_q^2 + q^2M^4\right)}{1 + 2q^2M^2 + 2q\left(M_q^2 + q^2M^4\right)^{1/2}}\left[\Gamma(1-\omega)F\left(\omega - \frac{1}{2}, \omega + \frac{1}{2}, 2\omega - 1, \frac{2}{1+\eta_q}\right)\right.$$
$$\left.\left. + \left(\frac{1}{2+2\eta_q}\right)^{2-2\omega}\frac{\Gamma(\omega-1)\Gamma(5/2-\omega)}{\Gamma(\omega+1/2)}F\left(\frac{3}{2} - \omega, \frac{5}{2} - \omega, 3 - 2\omega, \frac{2}{1+\eta_q}\right)\right]\right\}$$
$$+ \frac{g^2}{32\pi^{5/2}}\int_0^1 \mathrm{d}z\left\{-q\left(M_q^2 + q^2M^4\right)^{1/2}\left[\Gamma\left(\frac{3}{2} - \omega\right)F\left(\omega - 1, \omega - \frac{1}{2}, 2\omega - 2, \frac{2}{1+\eta_q}\right)\right.\right.$$
$$\left. + \left(\frac{1}{2+2\eta_q}\right)^{3-2\omega}\frac{\Gamma(5/2-\omega)\Gamma(\omega-3/2)}{\Gamma(\omega-1/2)}F\left(2 - \omega, \frac{5}{2} - \omega, 4 - 2\omega, \frac{2}{1+\eta_q}\right)\right]$$
$$+ \left(M_q^2 + q^2M^4\right)^{2-\omega}\left[(q^2)^{2-\omega}(q^2M_q^2 + q^4M^4)^{\omega/2-5/4}\Gamma\left(\frac{5}{2} - \omega\right)\right.$$
$$+ \left[1 + 2q^2M^2 + (1 + 4\sinh^2(qs^0)z(1-z))^{1/2}\right]^{\omega-3/2}\frac{2^{3/2-\omega}\Gamma\left(\frac{3}{2} - \omega\right)(q^2)^{2-\omega}}{[1 + 4\sinh^2(qs^0)z(1-z)]^{1/2}}$$
$$\left.\left.\times\left[1 - \left(\frac{1 + 2q^2\xi^2 - (1 + 4\sinh^2(qs^0)z(1-z))^{1/2}}{1 + 2q^2\xi^2 + (1 + 4\sinh^2(qs^0)z(1-z))^{1/2}}\right)^{\omega-3/2}\right]\right]\right\}. \tag{47}$$

Now we restrict the domain of this expression to a domain for which $\mathrm{Re}\,\omega < 1$, say to the strip $1/2 < \mathrm{Re}\,\omega < 1$. Finally, by taking the limit $q \to 0$ in this strip, we obtain the result

$$\lim_{q\to 0}\Gamma^{(4)}(m^2, s, \omega) = \frac{\mathrm{i}g^2}{32\pi^2}\Gamma(2-\omega)\int_0^1 \mathrm{d}z\left[\frac{4\pi\mu^2}{m^2 - s^2z(1-z)}\right]^{2-\omega}, \tag{48}$$

which can be trivially analytically continued to the whole complex ω-plane, except for simple poles at $\omega = 2, 3, \ldots$. This is the usual non-deformed expression for the vertex function, which diverges for even dimensions $2\omega = 4, 6, \ldots$, in particular for the physical dimension in consideration, $2\omega = 4$. Finally, we take this limit of physical dimension in the deformed vertex function (47) to obtain

$$\Gamma^{(4)}(m^2, s) = \mathrm{i}\frac{g^2}{16\pi^2}\int_0^1 \frac{\mathrm{d}z}{\left[1 + 2q^2M^2 + 2q\left(M_q^2 + q^2M^4\right)^{1/2}\right]^{3/2}}\left\{(1 + 2q^2M^2)\right.$$
$$\times(-\gamma + \log(2 + 2\eta_q))F\left(\frac{1}{2}, \frac{3}{2}, 1, \frac{2}{1+\eta_q}\right) + \frac{3/2q^2\left(M_q^2 + q^2M^4\right)}{1 + 2q^2M^2 + 2q\left(M_q^2 + q^2M^4\right)^{1/2}}$$
$$\times\left[(\gamma - 1 - 2\log(2 + 2\eta_q))F\left(\frac{3}{2}, \frac{5}{2}, 3, \frac{2}{1+\eta_q}\right) + \frac{16}{3}(1+\eta_q)^2 + \frac{8}{3}(1+\eta_q)\right.$$
$$\left.\left. + \frac{2}{\pi}\sum_{n=3}^{\infty}\frac{\Gamma\left(n - \frac{1}{2}\right)\Gamma\left(n + \frac{1}{2}\right)}{n!(n-2)!}(\Psi(n-1) + \gamma)\left(\frac{2}{1+\eta_q}\right)^{n-2}\right]\right\}$$

$$
\begin{aligned}
&+\frac{g^2}{16\pi^2}\int_0^1 \mathrm{d}z\Bigg\{-\frac{1}{\left[1+2q^2M^2+2q\left(M_q^2+q^2M^4\right)^{1/2}\right]^{1/2}}\\
&+\left(q^2M_q^2+q^4M^4\right)^{-1/4}+\left[\frac{1+2q^2M^2+(1+4\sinh^2(qs^0)z(1-z))^{1/2}}{1+4\sinh^2(qs^0)z(1-z)}\right]^{1/2}\\
&\times\left[1-\sqrt{\frac{1+2q^2M^2-(1+4\sinh^2(qs^0)z(1-z))^{1/2}}{1+2q^2M^2+(1+4\sinh^2(qs^0)z(1-z))^{1/2}}}\right]\Bigg\}, \qquad (49)
\end{aligned}
$$

which is finite as expected from the previous section. We should note that the finiteness of the deformed theory is guaranteed by the non-zero value of the fundamental length q. If we were considering q not as a fundamental length of the theory, but as a usual regularizing parameter it would be necessary to eliminate it at the end of the calculations by taking the limit $q \to 0$. In this case, this limit should give the same physical part of the vertex function modulo finite or divergent unphysical terms. That is indeed the case as can be seen from expressions (48) for small $\epsilon = 2 - \omega$ and (49) for small q, respectively,

$$
\begin{aligned}
\Gamma^{(4)}(m^2,s)|_{\epsilon\ll 1} &= \frac{\mathrm{i}g^2}{32\pi^2\varepsilon}-\frac{\mathrm{i}g^2\gamma}{32\pi^2}+\frac{\mathrm{i}g^2}{32\pi^2}\log\left(\frac{4\pi\mu^2}{m^2}\right)\\
&\quad -\frac{\mathrm{i}g^2}{32\pi^2}\int_0^1 \mathrm{d}z\log\left[1-\frac{s^2}{m^2}z(1-z)\right] \qquad (50)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma^{(4)}(m^2,s)|_{qm\ll 1} &= -\frac{\mathrm{i}g^2}{16\pi^2}\log(qm)+\frac{g^2}{16\pi^2(qm)^{1/2}}-\frac{\mathrm{i}g^2}{16\pi^2}(\gamma-2)+\frac{3g^2}{16\pi^2}\\
&\quad -\frac{\mathrm{i}g^2}{32\pi^2}\int_0^1 \mathrm{d}z\log\left[1-\frac{s^2}{m^2}z(1-z)\right]. \qquad (51)
\end{aligned}
$$

4. Conclusion

We have considered an effective model of a scalar field with quartic self-interaction and a κ-deformed dispersion relation to determine a possible regularizing effect of the deformation on the otherwise primitively divergent diagrams of the theory. The deformation parameter can be viewed as a fundamental length playing the role of a natural regularizing parameter, according to the original Heisenberg's proposal. In our model, it is easy to see that the deformation by itself is not sufficient to render the diagrams finite. However, it is found that the deformation leads to finite diagrams at the physical dimension after the usual analytic continuations are performed on dimensionally regularized diagrams. It is clear from our work that this result is a consequence of both deformation and dimensional regularization. Although the results of other regularization procedures applied to the deformed model are worth investigating, it seems quite natural that this particular deformation used in the model singles out this particular regularization procedure, the dimensional regularization, since both these mathematical structures deal with spacetime as their main object. Indeed, as mentioned, the κ-deformation has its origin in a deformed algebra of spacetime symmetries, while the dimensional regularization is based on the continuation of spacetime from the physical dimension to an arbitrary complex dimension. Of course, it is well known that regularization procedures can lead to finite results, and specially interesting examples are provided by analytic regularization methods (for a review see, e.g., [28]). However, in the present case, there seems to exist a natural connection between the two ingredients, the deformed symmetry and the complexification of spacetime, that lead to the finite results.

Their interplay to give the finite results is made explicit in the main results of the present work, which are expressions (25) and (46) for the self-energy and vertex diagrams as functions of the complex dimension variable 2ω. These expressions show that the poles of the deformed diagrams occur for odd dimensions, while the non-deformed diagrams have poles for even dimensions. This is an interesting result if we take into consideration that even dimensions as, e.g., 2, 4, 10 or 26 occur in important physical systems. The result shows explicitly that the singular dimensions of the deformed diagrams are the exact shifting of 1 unit to the right of the singular dimensions of the non-deformed diagrams, to wit: the singular dimensions for the deformed self-energy are $2\omega = 3, 5, 7, \ldots$ and for the deformed vertex $2\omega = 5, 7, \ldots$. It would be interesting to verify if the results of the present work in the context of an effective model with the LIV will also appear in theories with complete κ-deformed or Lorentz invariance [17, 19–21]. Let us finish this conclusion with one intuitive physical picture behind the finite expressions of the κ-deformed theory. It has been pointed out in early works on κ-deformed Poincaré algebra [11, 13] that the q-differential operator ∂_q which was defined at the beginning of section 2, and is the sole responsible for the κ-deformation of the Klein–Gordon equation, generates time translation in finite jumps. Actually, it acts on a function f of time as a finite difference operator with symmetrical shifts of size q along imaginary time, $\partial_q f(t) = [f(t+\mathrm{i}q) - f(t-\mathrm{i}q)]/(2q)$. Such a discretization of time evolution could be a reason for the observed regularization in a κ-deformed theory.

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